STRESS WAVES IN A VISCOELASTIC MEDIUM WITH A SINGULAR HEREDITARY KERNEL

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We consider stress waves in a viscoelastic medium with a singular hereditary kernel. It is shown that in such a medium, in contrast to the models of Maxwell and of a standard linear body, there is a propagation of waves on which the stresses vary continuously during the transition through the wave front.

Problems on the propagation of stress waves in semiinfinite viscoelastic bars were considered in [1-5], in which, using Laplace and Fourier integral transformations, solutions were obtained for the models of Maxwell [4], Voigt, and a standard linear body [5]. However, use of integral transformations causes definite computational difficulties, connected with the transition from the transform to the inverse transform; to eliminate these difficulties, approximate methods are frequently used: asymptotic formulas [6], expansions near the wave front [3], and also various approximations [7, 8].

Below we investigate stress waves in a viscoelastic bar, the hereditary properties of which are described by Boltzmann–Volterra relations with a singular hereditary kernel [9].

The stress $\sigma(x, t)$ in a viscoelastic semiinfinite bar with load $\sigma(0, t)$ given at the end has the form [10]

$$\sigma(x, t) = \frac{1}{2\pi i} \int\limits_{\alpha - i\infty}^{\alpha + i\infty} \sum(0, p) \exp\{-pxc_\infty^{-1}\lambda(p) + pt\}\, dp \tag{1}$$

$$\sum(0, p) = \int\limits_0^\infty \sigma(0, t) \exp(-pt)\, dt, \qquad \lambda(p) = [1 + \nu_\sigma K(p)]^{1/2}$$
$$\nu_\sigma = \Delta J J_\infty^{-1}, \quad \Delta J = J_0 - J_\infty, \quad c_\infty^{-2} = \rho J_\infty \tag{2}$$

where J_0 and J_∞ are, respectively, the nonrelaxation and relaxation values of the compliance, $K(p)$ is the Laplace transform of the aftereffect kernel, and ρ is the density of the medium.

We assume that the boundary stress $\sigma(0, t)$ is given by the Heaviside unit function $H(t)$:

$$\sigma(0, t) = \sigma_0 H(t), \quad \Sigma(0, p) = \sigma_0 p^{-1} \tag{3}$$

The problem of the propagation of a pulsed load in such a medium was considered in [11].

Substituting (3) into (1), we obtain

$$\sigma(x, t) = \frac{\sigma_0}{2\pi i} \int\limits_{\alpha - i\infty}^{\alpha + i\infty} p^{-1} \exp\{-p\lambda(p) xc_\infty^{-1} + pt\}\, dp \tag{4}$$

We consider as the aftereffect kernel the fractional-exponent function of Rabotnov [9]

$$E_\gamma(-1, t, s_\sigma) = t^{\gamma-1} \sum_{n=0}^\infty \frac{(-1)^n (t s_\sigma)^\gamma}{\Gamma[\gamma(n+1)]}, \qquad K(p) = \frac{s_\sigma^\gamma}{s_\sigma^\gamma + p^\gamma} \tag{5}$$

Here $s_{\sigma(\varepsilon)}$ is the frequency of retardation (relaxation), and γ is the divisibility parameter $(0 < \gamma \leq 1)$.

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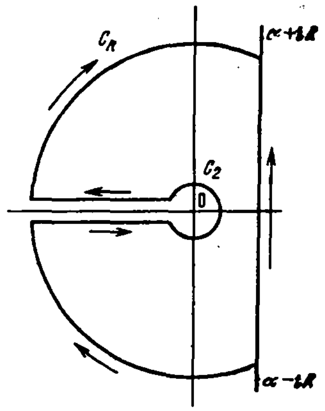

Fig. 1

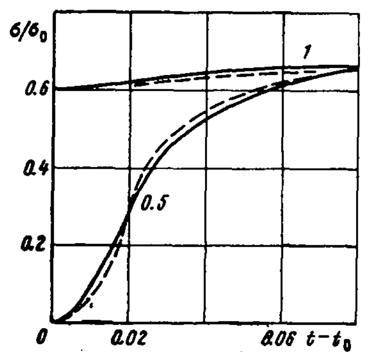

Fig. 2

Substituting (5) into (4), taking account of (2), we obtain

$$\sigma(x,t)=\frac{\sigma_0}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} p^{-1}\exp\left[-p\left(p^{\gamma}+s_{\varepsilon}^{\gamma}\right)^{1/2}\left(p^{\gamma}+s_{\sigma}^{\gamma}\right)^{-1/2}xc_{\infty}^{-1}+pt\right]dp \qquad (6)$$
$$(s_{\varepsilon}^{\gamma}s_{\sigma}^{\gamma}=J_0J_{\infty}^{-1})$$

The integrand in Eq. (6) for $\gamma\neq 1$ has a first-order pole at the point $p=0$ and branch points $p=0$ and $p=\infty$. The points $p_{1,2}=(-1)^{1/\gamma}s_{\varepsilon,\sigma}$ are not singularities, since they fall on the second sheet of a Riemann surface. The converse theorem is applicable to multivalued functions only for the first sheet of a Riemann surface ($|\arg p|<\pi$), and therefore the integration in Eq. (6) should be carried out over the contour shown in Fig. 1, for $R\to\infty$, $r\to 0$. Taking into account that the integral (6) is nonzero for the condition

$$\mathrm{Re}\left[pt-\lambda(p)\,xc_{\infty}^{-1}\right]\to-\infty,\ |p|\to\infty,\ {}^1\!/_2\pi<|\arg p|<\pi \qquad (7)$$

(the condition of passage of the wave through the point x), the expression for $\sigma(x, t)$ is written in the form

$$\sigma(x,t)=\sigma_0\left[1+\frac{1}{\pi}\int\limits_0^{\infty}\frac{1}{s}F(x,t,s)\,ds\right]H\left(t-\frac{x}{c_{\infty}}\right)$$
$$\left(F(x,t,s)=\exp\left\{sxc_{\infty}^{-1}R_1^{1/2}R_2^{-1/2}\cos\frac{\varphi_1-\varphi_2}{2}-st\right\}\sin\left[sxc_{\infty}^{-1}R_1^{1/2}R_2^{-1/2}\sin\frac{\varphi_1-\varphi_2}{2}\right]\right.$$
$$R_1^{2}=s_{\varepsilon}^{2\gamma}+2(ss_{\varepsilon})^{\gamma}\cos(\pi\gamma)+s^{2\gamma},\ R_2^{2}=s_{\sigma}^{2\gamma}+2(ss_{\sigma})^{\gamma}\cos(\pi\gamma)+s^{2\gamma} \qquad (8)$$
$$\left.\mathrm{tg}\,\varphi_1=\frac{\sin\pi\gamma}{s^{-\gamma}s_{\varepsilon}^{\gamma}+\cos\pi\gamma},\ \mathrm{tg}\,\varphi_2=\frac{\sin\pi\gamma}{s^{-\gamma}s_{\sigma}^{\gamma}+\cos\pi\gamma}\right)$$

Note that for $\gamma=1$ (the model of a standard linear body) the expression for the stress was investigated in [5].

In addition to the exact solution (8) there is interest in the expansion of $\sigma(x, t)$ near the wave front in a series in powers of $(t-t_0)^{\gamma}$. This series is obtained based on the "direct" method discussed in [7, 8]

$$\sigma(x,t)\approx\sigma_0\exp\left\{-{}^1\!/_2\gamma\left(s_{\varepsilon}^{\gamma}-s_{\sigma}^{\gamma}\right)t_0\left(t-t_0\right)^{\gamma-1}\right\}\quad(\gamma\neq 1) \qquad (9)$$

$$\sigma(x,t)\approx\sigma_0'\left[1+a_2t_0(t-t_0)+\left({}^1\!/_2a_2^2t_0-a_3\right)t_0(t-t_0)^2\right]\exp\{-a_1t_0\}\ (\gamma=1)$$
$$t_0=xc_{\infty}^{-1},\ a_1={}^1\!/_2(s_{\varepsilon}-s_{\sigma}),\ a_2={}^1\!/_8(s_{\varepsilon}-s_{\sigma})(s_{\varepsilon}+3s_{\sigma}) \qquad (10)$$
$$a_3={}^1\!/_{16}(s_{\varepsilon}-s_{\sigma})\left([s_{\varepsilon}+s_{\sigma}]^2+4s_{\sigma}^2\right)$$

The principal difference between expressions (7) and (8) consists in the fact that for fractional γ the stress is continuous near the wave front, unlike the model of a standard linear body, in which the stresses undergo a discontinuity.

Figure 2 gives results of the calculation of the stress $\sigma(x, t)$ using the integral (8) (solid lines) and expressions (9) and (10) (dashed lines) with $t_0=2$ for $s_{\varepsilon}/s_{\sigma}=1.5$. The values of the parameter γ are indicated by the numbers on the curves.

The calculations were made using a Mir-1 computer.

For fractional γ a wave propagates in the bar, the value of the stress being zero on the front and varying continuously in the transition through the wave front.

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